

Two positive solutions for a nonlinear four-point boundary value problem with a p -Laplacian operator *

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Abstract: In this paper, we study the existence of positive solutions for a nonlinear four-point boundary value problem with a p -Laplacian operator. By using a three functionals fixed point theorem in a cone, the existence of double positive solutions for the nonlinear four-point boundary value problem with a p -Laplacian operator is obtained. This is different than previous results.

Key words: p -Laplacian operator; Positive solution; Fixed point theorem; Four-point boundary value problem

1. Introduction

In this paper we are interested in the existence of positive solutions for the following nonlinear four-point boundary value problem with a p -Laplacian operator:

$$(\phi_p(u'))' + e(t)f(u(t)) = 0, \quad 0 < t < 1, \quad (1.1)$$

$$\mu\phi_p(u(0)) - \omega\phi_p(u'(\xi)) = 0, \quad \rho\phi_p(u(1)) + \tau\phi_p(u'(\eta)) = 0. \quad (1.2)$$

where $\phi_p(s)$ is a p -Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_q = (\phi_p)^{-1}$, $\frac{1}{q} + \frac{1}{p} = 1$, $\mu > 0$, $\omega \geq 0$, $\rho > 0$, $\tau \geq 0$, $\xi, \eta \in (0, 1)$ is prescribed and $\xi < \eta$, $e : (0, 1) \rightarrow [0, \infty)$, $f : [0, +\infty) \rightarrow [0, +\infty)$.

In recent years, because of the wide mathematical and physical background [1,2,12], the existence of positive solutions for nonlinear boundary value problems with p -Laplacian has received wide attention. There exists a very large number of papers devoted to the existence of solutions of the p -Laplacian operators with two or three-point boundary conditions, for example,

$$u(0) = 0, \quad u(1) = 0,$$

$$u(0) - B_0(u'(0)) = 0, \quad u(1) + B_1(u'(1)) = 0,$$

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$$\begin{aligned}u(0) - B_0(u'(0)) &= 0, \quad u'(1) = 0, \\u'(0) &= 0, \quad u(1) + B_1(u'(1)) = 0,\end{aligned}$$

and

$$\begin{aligned}u(0) &= 0, \quad u(1) = u(\eta), \\u(0) - B_0(u'(\eta)) &= 0, \quad u(1) + B_1(u'(1)) = 0, \\u(0) - B_0(u'(0)) &= 0, \quad u(1) + B_1(u'(\eta)) = 0, \\au(r) - bp(r)u'(r) &= 0, \quad cu(R) + dp(R)u'(R) = 0.\end{aligned}$$

For further knowledge, see [3-11,13]. The methods and techniques employed in these papers involve the use of Leray-Schauder degree theory [4], the upper and lower solution method [5], fixed point theorem in a cone [3,6-8,10,11,13], and the quadrature method [9]. However, there are several papers dealing with the existence of positive solutions for four-point boundary value problem [13-15,18].

Motivated by results in [14], this paper is concerned with the existence of two positive solutions of the boundary value problem (1.1)-(1.2). Our tool in this paper will be a new double fixed point theorem in a cone [11,16,17,19]. The result obtained in this paper is essentially different from the previous results in [14].

In the rest of the paper, we make the following assumptions:

(H1) $f \in C([0, +\infty), [0, +\infty))$;

(H2) $e(t) \in C((0, 1), [0, +\infty))$, and $0 < \int_0^1 e(t)dt < \infty$. Moreover, $e(t)$ does not vanish identically on any subinterval of $(0, 1)$.

Define

$$f_0 = \lim_{u \rightarrow 0^+} \frac{f(u)}{u^{p-1}}, \quad f_\infty = \lim_{u \rightarrow \infty} \frac{f(u)}{u^{p-1}}.$$

2. Some background definitions

In this section we provide some background material from the theory of cones in Banach space, and we state a two fixed point theorem due to Avery and Henderson [19].

If $P \subset E$ is a cone, we denote the order induced by P on E by \leq . That is

$$x \leq y \text{ if and only if } y - x \in P.$$

Definition 2.1 Given a cone P in a real Banach spaces E , a functional $\psi : P \rightarrow \mathbb{R}$ is said to be increasing on P , provided $\psi(x) \leq \psi(y)$, for all $x, y \in P$ with $x \leq y$.

Definition 2.2 Given a nonnegative continuous functional γ on a cone P of a real Banach space E (i.e., $\gamma : P \rightarrow [0, +\infty)$ continuous), we define, for each $d > 0$, the set

$$P(\gamma, d) = \{x \in P | \gamma(x) < d\}.$$

In order to obtain multiple positive solutions of (1.1)-(1.2), the following fixed point theorem of Avery and Henderson will be fundamental.

Theorem 2.1 [19] *Let P be a cone in a real Banach space E . Let α and γ be increasing, nonnegative continuous functional on P , and let θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some $c > 0$ and $M > 0$,*

$$\gamma(x) \leq \theta(x) \leq \alpha(x), \quad \text{and} \quad \|x\| \leq M\gamma(x)$$

for all $x \in \overline{P(\gamma, c)}$. Suppose there exist a completely continuous operator $\Phi : \overline{P(\gamma, c)} \rightarrow P$ and $0 < a < b < c$ such that

$$\theta(\lambda x) \leq \lambda \theta(x) \quad \text{for} \quad 0 \leq \lambda \leq 1 \quad \text{and} \quad x \in \partial P(\theta, b),$$

and

- (i) $\gamma(\Phi x) < c$, for all $x \in \partial P(\gamma, c)$,
- (ii) $\theta(\Phi x) > b$ for all $x \in \partial P(\theta, b)$,
- (iii) $P(\alpha, a) \neq \emptyset$ and $\alpha(\Phi x) < a$, for $x \in \partial P(\alpha, a)$.

Then Φ has at least two fixed points x_1 and x_2 belonging to $\overline{P(\gamma, c)}$ satisfying

$$a < \alpha(x_1) \quad \text{with} \quad \theta(x_1) < b,$$

and

$$b < \theta(x_2) \quad \text{with} \quad \gamma(x_2) < c.$$

3. Existence of two positive solutions of (1.1)-(1.2)

In this section, by defining an appropriate Banach space and cones, we impose growth conditions on f which allow us to apply the above two fixed point theorem in establishing the existence of double positive solutions of (1.1)-(1.2). Firstly, we mention without proof several fundamental results.

Lemma 3.1 [Lemma 2.1, 14]. *If condition (H2) holds, then there exists a constant $\delta \in (0, \frac{1}{2})$ that satisfies*

$$0 < \int_{\delta}^{1-\delta} e(t) dt < \infty.$$

Furthermore, the function:

$$y_1(t) = \int_{\delta}^t \phi_q \left(\int_s^t e(r) dr \right) ds + \int_t^{1-\delta} \phi_q \left(\int_t^s e(r) dr \right) ds, \quad t \in [\delta, 1-\delta],$$

is a positive continuous function on $[\delta, 1-\delta]$. Therefore $y_1(t)$ has a minimum on $[\delta, 1-\delta]$, so it follows that there exists $L_1 > 0$ such that

$$\min_{t \in [\delta, 1-\delta]} y_1(t) = L_1.$$

If $E = C[0, 1]$, then E is a Banach space with the norm $\|u\| = \sup_{t \in [0, 1]} |u(t)|$. We note that, from the nonnegativity of e and f , a solution of (1.1)-(1.2) is nonnegative and concave on $[0, 1]$. Define

$$P = \{u \in E : u(t) \geq 0, u(t) \text{ is concave function, } t \in [0, 1]\}.$$

Lemma 3.2 [Lemma 2.2, 14]. Let $u \in P$ and δ be as Lemma 3.1, then

$$u(t) \geq \delta \|u\|, \quad t \in [\delta, 1 - \delta].$$

Lemma 3.3 [Lemma 2.3, 14]. Suppose that conditions (H1), (H2) hold. Then $u(t) \in E \cap C^2(0, 1)$ is a solution of boundary value problem (1.1)-(1.2) if and only if $u(t) \in E$ is a solution of the following integral equation:

$$u(t) = \begin{cases} \phi_q\left(\frac{\omega}{\mu} \int_{\xi}^{\sigma} e(r) f(u(r)) dr\right) + \int_0^t \phi_q\left(\int_s^{\sigma} e(r) f(u(r)) dr\right) ds, & 0 \leq t \leq \sigma, \\ \phi_q\left(\frac{\tau}{\rho} \int_{\sigma}^{\eta} e(r) f(u(r)) dr\right) + \int_t^1 \phi_q\left(\int_{\sigma}^s e(r) f(u(r)) dr\right) ds, & \sigma \leq t \leq 1, \end{cases}$$

where $\sigma \in [\xi, \eta] \subset (0, 1)$ and $u'(\sigma) = 0$.

By means of the well known Guo-Krasnoselskii fixed point theorem in a cone, Su et al. [14] established the existence of at least one positive solution for (1.1)-(1.2) under some superlinear and sublinear assumptions imposed on the nonlinearity of f , which can be listed as

- (i) $f_0 = 0$ and $f_{\infty} = +\infty$ (superlinear), or
- (ii) $f_0 = +\infty$ and $f_{\infty} = 0$ (sublinear).

Using the same theorem, the authors also proved the existence of two positive solutions of (1.1)-(1.2) when f satisfies

- (iii) $f_0 = f_{\infty} = 0$, or
- (iv) $f_0 = f_{\infty} = +\infty$.

When $f_0, f_{\infty} \notin \{0, +\infty\}$, set

$$\theta^* = \frac{2}{L_1}, \quad \theta_* = \frac{1}{\left(1 + \phi_q\left(\frac{\omega}{\mu}\right)\right) \phi_q\left(\int_0^1 e(r) dr\right)},$$

and in the following, always assume δ be as in Lemma 3.1, the existence of double positive solutions of boundary value problem (1.1)-(1.2) can be list as follows:

Theorem 3.1 [Theorem 4.3, 14]. Suppose that conditions (H1),(H2) hold. Also assume that f satisfies

- (A1) $f_0 = \lambda_1 \in \left[0, \left(\frac{\theta_*}{4}\right)^{p-1}\right)$;
- (A2) $f_{\infty} = \lambda_2 \in \left[0, \left(\frac{\theta_*}{4}\right)^{p-1}\right)$;
- (A3) $f(u) \leq (MR)^{p-1}, 0 \leq u \leq R$,

where $M \in (0, \theta_*)$. Then the boundary value problem (1.1)-(1.2) has at least two positive solutions u_1, u_2 such that

$$0 < \|u_1\| < R < \|u_2\|.$$

Theorem 3.2 [Theorem 4.4, 14]. Suppose that conditions (H1),(H2) hold. Also assume that f satisfies

- (A4) $f_0 = \lambda_1 \in \left[\left(\frac{2\theta^*}{\delta}\right)^{p-1}, \infty\right)$;

$$(A5) \quad f_\infty = \lambda_2 \in \left[\left(\frac{2\theta^*}{\delta} \right)^{p-1}, \infty \right);$$

$$(A6) \quad f(u) \geq (mr)^{p-1}, \delta r \leq u \leq r,$$

where $m \in (\theta^*, \infty)$. Then the boundary value problem (1.1)-(1.2) has at least two positive solutions u_1, u_2 such that

$$0 < \|u_1\| < r < \|u_2\|.$$

When we see such a fact, we cannot but ask “Whether or not we can obtain a similar conclusion if neither $f_0 \in [(\frac{2\theta^*}{\delta})^{p-1}, \infty)$ nor $f_0 \in [0, (\frac{\theta^*}{4})^{p-1}]$.” Motivated by the above mentioned results, in this paper, we attempt to establish simple criteria for the existence of at least two positive solutions of (1.1)-(1.2). Our result is based on Theorem 2.1 and gives a positive answer to the question stated above.

Set

$$y_2(t) := \phi_q \left(\int_\delta^t e(r) dr \right) + \phi_q \left(\int_t^{1-\delta} e(r) dr \right), \quad \delta \leq t \leq 1 - \delta.$$

For notational convenience, we introduce the following constants:

$$L_2 = \min_{\delta \leq t \leq 1-\delta} y_2(t),$$

and

$$L_3 = \delta \phi_q \left(\int_0^1 e(r) dr \right) + \max \left\{ \phi_q \left(\frac{\omega}{\mu} \int_\xi^\eta e(r) dr \right), \phi_q \left(\frac{\tau}{\rho} \int_\xi^\eta e(r) dr \right) \right\},$$

$$Q = \phi_q \left(\int_0^1 e(r) dr \right) + \max \left\{ \phi_q \left(\frac{\omega}{\mu} \int_\xi^\eta e(r) dr \right), \phi_q \left(\frac{\tau}{\rho} \int_\xi^\eta e(r) dr \right) \right\}.$$

Finally, we define the nonnegative, increasing continuous functions γ, θ and α by

$$\gamma(u) = \min_{t \in [\delta, 1-\delta]} u(t),$$

$$\theta(u) = \frac{1}{2} [u(\delta) + u(1-\delta)], \quad \alpha(u) = \max_{0 \leq t \leq 1} u(t).$$

We observe here that, for every $u \in P$,

$$\gamma(u) \leq \theta(u) \leq \alpha(u).$$

It follows from Lemma 3.2 that, for each $u \in P$, one has $\gamma(u) \geq \delta \|u\|$, so $\|u\| \leq \frac{1}{\delta} \gamma(u)$, for all $u \in P$. We also note that $\theta(\lambda u) = \lambda \theta(u)$, for $0 \leq \lambda \leq 1$, and $u \in \partial P(\theta, b)$.

The main result of this paper is as follows:

Theorem 3.3 Assume that (H_1) and (H_2) hold, and suppose that there exist positive constants $0 < a < b < c$ such that $0 < a < \delta b < \frac{\delta^2 L_2}{2L_3} c$, and f satisfies the following conditions

$$(D1) \quad f(v) < \phi_p \left(\frac{a}{Q} \right), \quad \text{if } 0 \leq v \leq a;$$

$$(D2) \quad f(v) > \phi_p \left(\frac{2b}{\delta L_2} \right), \quad \text{if } \delta b \leq v \leq \frac{b}{\delta};$$

$$(D3) \quad f(v) < \phi_p \left(\frac{c}{L_3} \right), \quad \text{if } 0 \leq v \leq \frac{c}{\delta};$$

Then, the boundary value problem (1.1) and (1.2) has at least two positive solutions u_1 and u_2 such that

$$a < \max_{t \in [0,1]} u_1(t), \quad \text{with } \frac{1}{2} [u_1(\delta) + u_1(1-\delta)] < b;$$

and

$$b < \frac{1}{2}[u_2(\delta) + u_2(1 - \delta)], \quad \text{with} \quad \min_{t \in [\delta, 1 - \delta]} u_2(t) < c.$$

Proof. We define the operator: $\Phi : P \rightarrow P$,

$$(\Phi u)(t) := \begin{cases} \phi_q\left(\frac{\omega}{\mu} \int_{\xi}^{\sigma} e(r)f(u(r))dr\right) + \int_0^t \phi_q\left(\int_s^{\sigma} e(r)f(u(r))dr\right)ds, & 0 \leq t \leq \sigma, \\ \phi_q\left(\frac{\tau}{\rho} \int_{\sigma}^{\eta} e(r)f(u(r))dr\right) + \int_t^1 \phi_q\left(\int_{\sigma}^s e(r)f(u(r))dr\right)ds, & \sigma \leq t \leq 1, \end{cases}$$

for each $u \in P$, where $\sigma \in [\xi, \eta] \subset (0, 1)$. It is shown in Lemma 3.3 that the operator $\Phi : P \rightarrow P$ is well defined with $\|\Phi u\| = \Phi u(\sigma)$. In particular, if $u \in P(\gamma, c)$, we also have $\Phi u \in P$, moreover, a standard argument shows that $\Phi : P \rightarrow P$ is completely continuous (see [Lemma 2.4, 14]) and each fixed point of Φ in P is a solution of (1.1)-(1.2).

We now show that the conditions of Theorem 2.1 are satisfied.

To fulfill property (i) of Theorem 2.2, we choose $u \in \partial P(\gamma, c)$, thus $\gamma(u) = \min_{t \in [\delta, 1 - \delta]} u(t) = c$. Recalling that $\|u\| \leq \frac{1}{\delta}\gamma(u) = \frac{c}{\delta}$, we have

$$0 \leq u(t) \leq \|u\| \leq \frac{1}{\delta}\gamma(u) = \frac{c}{\delta}, \quad 0 \leq t \leq 1.$$

Then assumption (D3) of Theorem 3.2 implies

$$f(u(t)) < \phi_p\left(\frac{c}{L_3}\right), \quad 0 \leq t \leq 1.$$

(i) If $\sigma \in (0, \delta)$, we have

$$\begin{aligned} \gamma(\Phi u) &= \min_{t \in [\delta, 1 - \delta]} (\Phi u)(t) = (\Phi u)(1 - \delta) \\ &= \phi_q\left(\frac{\tau}{\rho} \int_{\sigma}^{\eta} e(r)f(u(r))dr\right) + \int_{1 - \delta}^1 \phi_q\left(\int_{\sigma}^s e(r)f(u(r))dr\right)ds \\ &\leq \phi_q\left(\frac{\tau}{\rho} \int_{\xi}^{\eta} e(r)f(u(r))dr\right) + \int_{1 - \delta}^1 \phi_q\left(\int_0^1 e(r)f(u(r))dr\right)ds \\ &\leq \left[\phi_q\left(\frac{\tau}{\rho} \int_{\xi}^{\eta} e(r)dr\right) + \delta\phi_q\left(\int_0^1 e(r)dr\right)\right] \cdot \frac{c}{L_3} < c. \end{aligned}$$

(ii) If $\sigma \in [\delta, 1 - \delta]$, we have

$$\begin{aligned} \gamma(\Phi u) &= \min_{t \in [\delta, 1 - \delta]} (\Phi u)(t) = \min\{(\Phi u)(\delta), (\Phi u)(1 - \delta)\} \\ &= \min\left\{\phi_q\left(\frac{\omega}{\mu} \int_{\xi}^{\sigma} e(r)f(u(r))dr\right) + \int_0^{\delta} \phi_q\left(\int_s^{\sigma} e(r)f(u(r))dr\right)ds, \right. \\ &\quad \left.\phi_q\left(\frac{\tau}{\rho} \int_{\sigma}^{\eta} e(r)f(u(r))dr\right) + \int_{1 - \delta}^1 \phi_q\left(\int_{\sigma}^s e(r)f(u(r))dr\right)ds\right\} \\ &\leq \max\left\{\phi_q\left(\frac{\omega}{\mu} \int_{\xi}^{\eta} e(r)f(u(r))dr\right) + \int_0^{\delta} \phi_q\left(\int_0^1 e(r)f(u(r))dr\right)ds, \right. \\ &\quad \left.\phi_q\left(\frac{\tau}{\rho} \int_{\xi}^{\eta} e(r)f(u(r))dr\right) + \int_{1 - \delta}^1 \phi_q\left(\int_0^1 e(r)f(u(r))dr\right)ds\right\} \\ &< \left[\max\left\{\phi_q\left(\frac{\omega}{\mu} \int_{\xi}^{\eta} e(r)dr\right), \phi_q\left(\frac{\tau}{\rho} \int_{\xi}^{\eta} e(r)dr\right)\right\} + \delta\phi_q\left(\int_0^1 e(r)dr\right)\right] \cdot \frac{c}{L_3} \\ &= c. \end{aligned}$$

(iii) If $\sigma \in (1 - \delta, 1)$, we have

$$\begin{aligned}
\gamma(\Phi u) &= \min_{t \in [\delta, 1-\delta]} (\Phi u)(t) = \Phi u(\delta) \\
&= \phi_q \left(\frac{\omega}{\mu} \int_{\xi}^{\sigma} e(r) f(u(r)) dr \right) + \int_0^{\delta} \phi_q \left(\int_s^{\sigma} e(r) f(u(r)) dr \right) ds \\
&\leq \phi_q \left(\frac{\omega}{\mu} \int_{\xi}^{\eta} e(r) f(u(r)) dr \right) + \int_0^{\delta} \phi_q \left(\int_0^1 e(r) f(u(r)) dr \right) ds \\
&\leq \left[\phi_q \left(\frac{\omega}{\mu} \int_{\xi}^{\eta} e(r) dr \right) + \delta \phi_q \left(\int_0^1 e(r) dr \right) \right] \cdot \frac{c}{L_3} \\
&< c.
\end{aligned}$$

Therefore, condition (i) of Theorem 2.2 is satisfied.

We next address (ii) of Theorem 2.2. For this, we choose $u \in \partial P(\theta, b)$ so that $\theta(u) = \frac{1}{2}[u(\delta) + u(1 - \delta)] = b$. Noting that

$$\|u\| \leq (1/\delta)\gamma(u) \leq (1/\delta)\theta(u) = b/\delta,$$

we have

$$\delta b < \delta \|u\| \leq u(t) \leq \frac{b}{\delta}, \quad \text{for } t \in [\delta, 1 - \delta].$$

Then (D2) yields

$$f(u(t)) > \phi_p \left(\frac{2b}{\delta L_2} \right), \quad \text{for } t \in [\delta, 1 - \delta].$$

As $\Phi u \in P$:

(i) If $\sigma \in (0, \delta)$, we have

$$\begin{aligned}
\theta(\Phi u) &= \frac{1}{2}(\Phi u(\delta) + \Phi u(1 - \delta)) \geq \Phi u(1 - \delta) \\
&= \phi_q \left(\frac{\tau}{\rho} \int_{\sigma}^{\eta} e(r) f(u(r)) dr \right) + \int_{1-\delta}^1 \phi_q \left(\int_{\sigma}^s e(r) f(u(r)) dr \right) ds \\
&\geq \int_{1-\delta}^1 \phi_q \left(\int_{\sigma}^{1-\delta} e(r) f(u(r)) dr \right) ds \\
&\geq \int_{1-\delta}^1 \phi_q \left(\int_{\delta}^{1-\delta} e(r) f(u(r)) dr \right) ds \\
&= \delta \phi_q \left(\int_{\delta}^{1-\delta} e(r) f(u(r)) dr \right) \\
&\geq \delta \phi_q \left(\int_{\delta}^{1-\delta} e(r) dr \right) \cdot \frac{2b}{\delta L_2} \geq 2b > b.
\end{aligned}$$

(ii) If $\sigma \in [\delta, 1 - \delta]$, we have

$$\begin{aligned}
2\theta(\Phi u) &= [\Phi u(\delta) + \Phi u(1 - \delta)] \\
&\geq \int_0^{\delta} \phi_q \left(\int_s^{\sigma} e(r) f(u(r)) dr \right) ds + \int_{1-\delta}^1 \phi_q \left(\int_{\sigma}^s e(r) f(u(r)) dr \right) ds \\
&\geq \int_0^{\delta} \phi_q \left(\int_{\delta}^{\sigma} e(r) f(u(r)) dr \right) ds + \int_{1-\delta}^1 \phi_q \left(\int_{\sigma}^{1-\delta} e(r) f(u(r)) dr \right) ds \\
&= \delta \left[\phi_q \left(\int_{\delta}^{\sigma} e(r) f(u(r)) dr \right) + \phi_q \left(\int_{\sigma}^{1-\delta} e(r) f(u(r)) dr \right) \right] \\
&\geq \delta \left[\phi_q \left(\int_{\delta}^{\sigma} e(r) dr \right) + \phi_q \left(\int_{\sigma}^{1-\delta} e(r) dr \right) \right] \cdot \frac{2b}{\delta L_2} \\
&\geq 2b.
\end{aligned}$$

(iii) If $\sigma \in (1 - \delta, 1)$, we have

$$\begin{aligned}
\theta(\Phi u) &= \frac{1}{2}(\Phi u(\delta) + \Phi u(1 - \delta)) \geq \Phi u(\delta) \\
&= \phi_q\left(\frac{\omega}{\mu} \int_{\xi}^{\sigma} e(r)f(u(r))dr\right) + \int_0^{\delta} \phi_q\left(\int_s^{\sigma} e(r)f(u(r))dr\right)ds \\
&\geq \int_0^{\delta} \phi_q\left(\int_{\delta}^{1-\delta} e(r)f(u(r))dr\right)ds \\
&> \delta \phi_q\left(\int_{\delta}^{1-\delta} e(r)dr\right) \cdot \frac{2b}{\delta L_2} \geq 2b > b.
\end{aligned}$$

Hence, condition (ii) of Theorem 2.2 holds.

To fulfill property (iii) of Theorem 2.2, we note $u_*(t) \equiv a/2, 0 \leq t \leq 1$, is a member of $P(\alpha, a)$ and $\alpha(u_*) = a/2$, so $P(\alpha, a) \neq 0$. Now, choose $u \in \partial P(\alpha, a)$, so that $\alpha(u) = \max_{t \in [0,1]} u(t) = a$ and implies $0 \leq u(t) \leq a, 0 \leq t \leq 1$. It follows from assumption (D1), $f(u(t)) \leq \phi_p(a/Q), t \in [0, 1]$. As before we obtain

$$\begin{aligned}
\alpha(\Phi u) &= \|\Phi u\| = \Phi u(\sigma) \\
&= \phi_q\left(\frac{\omega}{\mu} \int_{\xi}^{\sigma} e(r)f(u(r))dr\right) + \int_0^{\sigma} \phi_q\left(\int_s^{\sigma} e(r)f(u(r))dr\right)ds \\
&= \phi_q\left(\frac{\tau}{\rho} \int_{\sigma}^{\eta} e(r)f(u(r))dr\right) + \int_{\sigma}^1 \phi_q\left(\int_{\sigma}^s e(r)f(u(r))dr\right)ds \\
&\leq \max\left\{\phi_q\left(\frac{\omega}{\mu} \int_{\xi}^{\eta} e(r)dr\right) + \phi_q\left(\int_0^1 e(r)dr\right), \right. \\
&\quad \left. \phi_q\left(\frac{\tau}{\rho} \int_{\xi}^{\eta} e(r)dr\right) + \phi_q\left(\int_0^1 e(r)dr\right)\right\} \cdot \frac{a}{Q} \\
&\leq a.
\end{aligned}$$

Thus, condition (iii) of Theorem 2.1 is also satisfied. Consequently, an application of Theorem 2.1 completes the proof. \square

Finally, we present an example to explain our result.

Example. Consider the boundary value problem (1.1)-(1.2) with

$$p = \frac{3}{2}, \mu = 2, \rho = \omega = 1, \xi = \frac{1}{4}, \eta = \frac{1}{2}, \tau = 1, \delta = \frac{1}{4}, e(t) = t^{-\frac{1}{2}},$$

and

$$f(u) = \begin{cases} \frac{6\sqrt{2}u}{u+1}, & 0 \leq u \leq 200, \\ \frac{40}{67} + \frac{1202}{335}(u-200), & 200 \leq u \leq 250, \\ 180, & 250 < u, \end{cases}$$

Then (1.1)-(1.2) has at least two positive solutions.

Proof. In this example we have

$$\begin{aligned}
L_1 &= \min_{1/4 \leq x \leq 3/4} \left\{ \int_{1/4}^x \phi_q\left(\int_s^x t^{-1/2}dt\right)ds + \int_x^{3/4} \phi_q\left(\int_x^s t^{-1/2}dt\right)ds \right\} = \frac{3\sqrt{3}-5}{9}, \\
L_2 &= \min_{1/4 \leq x \leq 3/4} \left(\phi_q\left(\int_{1/4}^x t^{-1/2}dt\right) + \phi_q\left(\int_x^{3/4} t^{-1/2}dt\right) \right) = 2 - \sqrt{3}, \\
L_3 &= \delta \phi_q\left(\int_0^1 e(r)dr\right) + \max\left\{\phi_q\left(\frac{\omega}{\mu} \int_{\xi}^{\eta} e(r)dr\right), \phi_q\left(\frac{\tau}{\rho} \int_{\xi}^{\eta} e(r)dr\right)\right\} = 4 - 2\sqrt{2},
\end{aligned}$$

$$Q = \phi_q\left(\int_0^1 e(r)dr\right) + \max\left\{\phi_q\left(\frac{\omega}{\mu}\int_\xi^\eta e(r)dr\right), \phi_q\left(\frac{\tau}{\rho}\int_\xi^\eta e(r)dr\right)\right\} = 7 - 2\sqrt{2}.$$

Let $a = 80, b = 1000, c = 40000$. Then we have

$$f(u) = \frac{6\sqrt{2}u}{u+1} < \phi_p(a/Q), \quad \text{for } 0 \leq u \leq 80,$$

$$f(u) = 180 > \phi_p((2b)/(\delta L_2)), \quad \text{for } 250 \leq u \leq 4000,$$

$$f(u) = 180 < \phi_p(c/L_3), \quad \text{for } 0 \leq u \leq 160000.$$

Therefore, by Theorem 3.3 we deduce that (1.1)-(1.2) has at least two positive solutions u_1 and u_2 satisfying

$$80 < \max_{t \in [0,1]} u_1(t), \quad \text{with } \frac{1}{2}[u_1(\delta) + u_1(1-\delta)] < 1000;$$

and

$$1000 < \frac{1}{2}[u_2(\delta) + u_2(1-\delta)], \quad \text{with } \min_{t \in [\delta, 1-\delta]} u_2(t) < 40000.$$

Remark. We notice that in the above example, $f_0 = 6\sqrt{2} \approx 8.48528, (\frac{\theta_*}{4})^{p-1} = \frac{\sqrt{5}}{10} \approx 0.223607$ and $(\frac{2\theta_*}{\delta})^{p-1} = 6\sqrt{10+6\sqrt{3}} \approx 27.0947$. Therefore, Theorem 3.1 and Theorem 3.2 are not applicable to this example since conditions (A1) and (A4) fail.

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